W-2/3

# EECS 16B Section 3B

**Main Topic:** Change of Basis and Vector Differential Equations

**Administrivia:**

- HW 3 due Fri, 2/5
- Anonymous Feedback:
  bit.ly/maxwell-16B-feedback-sp21

**Agenda:**

- (a-c) Systems of Differential Equations
- (d-f) Direct Substitution
- Diagonalization
- Change of Basis
- (d-f) Matrices and Vectors
- Recap Diagram

**1. Changing Coordinates and Systems of Differential Equations**

Suppose we have the pair of differential equations (valid for $t \geq 0$)

$$\frac{d}{dt}x_1(t) = -9x_1(t)$$

$$\frac{d}{dt}x_2(t) = -2x_2(t)$$

with initial conditions $x_1(0) = -1$ and $x_2(0) = 3$.

(a) Solve for $x_1(t)$ and $x_2(t)$ for $t \geq 0$.

$\vec{x} = Ke^{\lambda t}$

$\lambda =$ coeff.

$K =$ scaling

$x_1(t) = K_1 e^{-9t}$

$x_2(t) = K_2 e^{-2t}$

$x_1(0) = -1 = K_1 e^0$

$-1 = K_1(1)$

$-1 = K_1$

$x_2(0) = 3 = K_2 e^0$

$3 = K_2$

$$\boxed{\begin{aligned} x_1(t) &= -e^{-9t} \\ x_2(t) &= 3e^{-2t} \end{aligned}}$$

**Suppose we are actually interested in a different set of variables with the following differential equations:**

"Coupled" System

$$\left\{ \begin{aligned} \frac{d}{dt} z_1(t) &= -5z_1(t) + 2z_2(t) \\ \frac{d}{dt} z_2(t) &= 6z_1(t) - 6z_2(t). \end{aligned} \right\}$$

(b) Write out the above system of differential equations in matrix form. Assuming that the initial state $\vec{z(0)} = \begin{bmatrix} 7 & 7 \end{bmatrix}^T$, can we solve this system directly?

$$\frac{d}{dt}\left[\vec{z}(t)\right] = \left[\ A\ \right]\left[\vec{z}(t)\right]$$

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \overbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}^{A}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

Can we find $z_1(t)$, $z_2(t)$?

(c) Consider that in our frustration with the previous system of differential equations, we start hearing voices. These voices whisper to us that that we should try the following change of variables:

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

"substitution"

Write out this transformation in matrix form ($\vec{z} = V\vec{y}$).

$$\vec{z} = \overbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}^{V}\vec{y}$$

$$\vec{y} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}^{-1}\vec{z}$$

$$= V^{-1}\vec{z}$$

# Method 1: Direct Substitution

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

(d) How do the initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$?

$z_1(0) = 7$

$z_2(0) = 7$

Solve for $y_1(0)$, $y_2(0)$?

$$\begin{cases} 7 = -y_1(0) + 2y_2(0) \\ 7 = 2y_1(0) + 3y_2(0) \end{cases}$$

$y_1(0) = -1$

$y_2(0) = 3$

(e) Rewrite the differential equations in terms of $y_i(t)$.
Can we solve this system of differential equations?

$$\begin{cases} y_1(t) = -\frac{3}{7} z_1(t) + \frac{2}{7} z_2(t) \\ y_2(t) = \frac{2}{7} z_1(t) + \frac{1}{7} z_2(t) \end{cases}$$

$$\begin{cases} \frac{d}{dt} y_1 = -\frac{3}{7}\frac{d}{dt} z_1 + \frac{2}{7}\frac{d}{dt} z_2 \\ \frac{d}{dt} y_2 = \frac{2}{7}\frac{d}{dt} z_1 + \frac{1}{7}\frac{d}{dt} z_2 \end{cases}$$

$$\frac{d}{dt} y_1 = -\frac{3}{7} \frac{d}{dt} z_1 + \frac{2}{7} \frac{d}{dt} z_2$$

$$\frac{d}{dt} y_2 = \frac{2}{7} \frac{d}{dt} z_1 + \frac{1}{7} \frac{d}{dt} z_2$$

$$\frac{d}{dt} z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt} z_2(t) = 6z_1(t) - 6z_2(t).$$

$$\frac{d}{dt} y_1 = -\frac{3}{7}(-5z_1 + 2z_2) + \frac{2}{7}(6z_1 - 6z_2)$$

$$\frac{d}{dt} y_2 = \frac{2}{7}(-5z_1 + 2z_2) + \frac{1}{7}(6z_1 - 6z_2)$$

$$\frac{d}{dt} y_1 = -9y_1$$
$$\frac{d}{dt} y_2 = -2y_2$$

(f) What are the solutions for $z_i(t)$?

$$y_1 = -e^{-9t}$$
$$\vec{y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$y_2 = 3e^{-2t}$$

$$z_1 = -y_1 + 2y_2 = e^{-9t} + 6e^{-2t}$$

$$z_2 = 2y_1 + 3y_2 = -2e^{-9t} + 9e^{-2t}$$

# Diagonalization

$A\vec{v} = \lambda\vec{v}$

WLOG, consider an $n \times n$ matrix $A$ with $n$ linearly-independent eigenvalue / eigenvector pairs $(\lambda_i, \vec{v}_i)$.
(Without Loss of Generality)

Capital $\lambda$

$$\Lambda = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad V = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

From 16A / 54, we know $A\vec{v}_i = \lambda_i \vec{v}_i$

$$\hookrightarrow AV = V\Lambda$$

Since our eigenvectors are linearly-independent, $V$ is invertible

$$AV = V\Lambda \rightarrow (A\cancel{V})\cancel{V^{-1}} = (V\Lambda)V^{-1} \rightarrow \boxed{A = V\Lambda V^{-1}}$$

"Eigendecomposition" aka "Diagonalization" of $A$

"Why should we care?"

- "Break apart" $A$ in terms of eigenvalues / eigenvectors
- Easy Matrix Exponentiation
- Develops idea of "Eigenbasis"

$$\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} = \overset{V}{\begin{bmatrix} \ \end{bmatrix}} \overset{\Lambda}{\begin{bmatrix} \ \end{bmatrix}} \overset{V^{-1}}{\begin{bmatrix} \ \end{bmatrix}}$$

# Change of Basis

"What is a basis?" = Set of vectors that "define" a vector space $V$ s.t. every vector $\vec{v} \in V$ can be written as a linear combination of the basis vectors

$$B = \{\vec{b}_1, \vec{b}_2\}$$

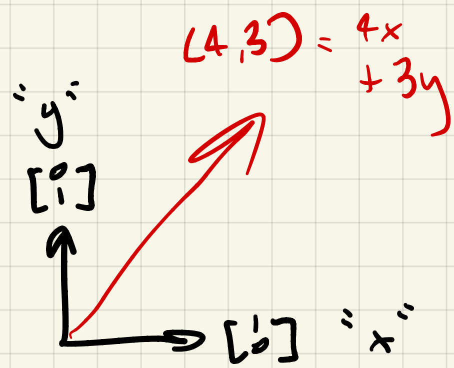
$\forall \vec{v} \in V$,

$$\vec{v} = \alpha \vec{b}_1 + \beta \vec{b}_2$$

= "Our coordinate system"

## Standard Basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$




2D Plane

## "Can we use a different basis?"

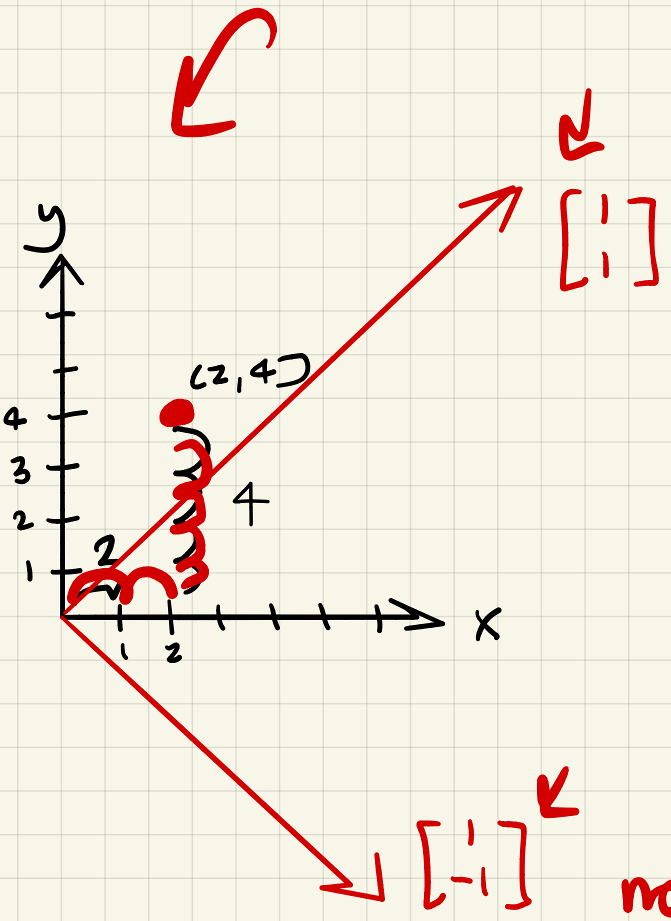
Analogy: Counting with different unit steps

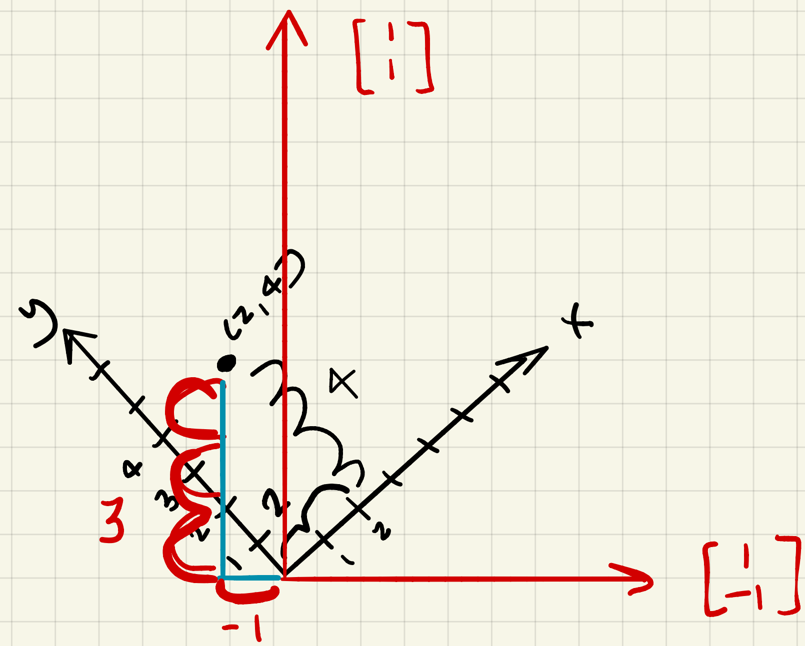
ex Concept of "8"

- Units of 1? $[1, 2, 3, 4, \ldots, 8] = 8$ steps $= \frac{8}{1} = 1^{-1} \cdot 8$
- Units of 2? $[2, 4, 6, 8] = 4$ steps $= \frac{8}{2} = 2^{-1} \cdot 8$
- Units of 4? $[4, 8] = 2$ steps $= \frac{8}{4} = 4^{-1} \cdot 8$

Let our unit be $U$. Then, $8_{new} = U^{-1}\, 8_{old}$

(ex) $\vec{z} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ in standard basis. What is $\vec{w}$, $\vec{z}$'s representation in the basis / coordinate system $P = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ ?




$(-1, 3) \to \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

new old

$$\vec{w} = P^{-1} \vec{z}$$

$$P \vec{w} = \vec{z}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

# Method 2: Matrices and Vectors

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

$$\vec{z} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \vec{y}$$

$$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

(d) How do the initial conditions for $z_i(t)$ translate into the initial conditions for $y_i(t)$?

$$\vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\vec{z} = V\vec{y}$$
$$\vec{y} = V^{-1}\vec{z}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-7}\begin{bmatrix} 3 & -2 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(e) Rewrite the differential equations in terms of $y_i(t)$.
Can we solve this system of differential equations?

$$\frac{d}{dt}z_1(t) = -5z_1(t) + 2z_2(t)$$
$$\frac{d}{dt}z_2(t) = 6z_1(t) - 6z_2(t).$$

$$z_1(t) = -y_1(t) + 2y_2(t)$$
$$z_2(t) = 2y_1(t) + 3y_2(t).$$

$\vec{z} = V\vec{y}$

$\vec{y} = V^{-1}\vec{z}$

$V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$

$A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$

$\frac{d}{dt}\vec{z} = A\vec{z}$

= Coupled diff. eq.

$\frac{d}{dt}\vec{y} = ?$

$\vec{y} = V^{-1}z$

$$\frac{d}{dt}\vec{y} = \frac{d}{dt}(V^{-1}\vec{z})$$
$$= V^{-1}\underbrace{\frac{d}{dt}\vec{z}}$$
$$= V^{-1}A\underbrace{\vec{z}}$$

$$\frac{d}{dt}\vec{y} = V^{-1}AV\underbrace{\vec{y}}$$

$$\frac{d}{dt}\vec{y} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}\vec{y}$$

Change of Basis

= Uncoupled

$$\frac{d}{dt}\vec{y}_1 = -9\vec{y}_1$$

$$\frac{d}{dt}\vec{y}_2 = -2\vec{y}_2$$

(f) What are the solutions for $z_i(t)$?

$$\vec{y}_1 = -e^{-9t}$$

$$\vec{y}_2 = 3e^{-2t}$$

$$\vec{z} = V\vec{y}$$

$$\vec{z} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$$

$V$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

# Process:

1. Find coefficient matrix $A$
2. Solve for $A = V \Lambda V^{-1}$
3. $\vec{y} = V^{-1}\vec{z}$ ; $\vec{z} = V\vec{y}$
4. $\frac{d}{dt}\vec{z} = A\vec{z} = V\Lambda V^{-1}\vec{z}$

$$\frac{d}{dt}V^{-1}\vec{z} = \Lambda V^{-1}\vec{z}$$

$$\boxed{\frac{d}{dt}\vec{y} = \Lambda\vec{y}}$$

"Decoupled Differential Equations"

5. $\vec{y}(0) = V^{-1}\vec{z}(0)$
6. Solve for $\vec{y}$
7. $\vec{z} = V\vec{y}$

$$\begin{tikzcd}
\vec{z} \arrow[r, "A"] \arrow[d, bend right, "V^{-1}"'] & \vec{y} \arrow[d, bend right, "V^{-1}"'] \\
\vec{z}\,' \arrow[r, "\Lambda"] \arrow[u, bend right, "V"'] & \vec{y}\,' \arrow[u, bend right, "V"']
\end{tikzcd}$$

$$\vec{z}\,' = V^{-1}\vec{z}$$

$$\vec{y}\,' = \Lambda V^{-1}\vec{z}$$

$$\vec{y} = V\Lambda V^{-1}\vec{z}$$